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LETTER TO THE EDITOR

Renormalisation on Sierpinski-type fractals

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Abstract. We present a family of deterministic fractals which generalise the d-dimensional Sierpinski gaskets and we establish their order of ramification and their fractal and spectral dimensions. Random walks on these fractals are renormalisable and lead to rational, not necessarily polynomial, mappings.

Fractal structures, both deterministic as the Sierpinski gaskets, and stochastic as the percolation clusters at criticality, have currently been a subject of intensive investigation (Mandelbrot 1982, Alexander and Orbach 1982, Gefen *et al* 1980, 1984). Applications include linear and branched polymers (Havlin and Ben-Avraham 1982), epoxy resins (Alexander *et al* 1983), amorphous and porous materials (Pfeifer and Avnir 1983, Klafter and Blumen 1984) and aggregates constructed by diffusion-limited growth (Witten and Sander 1981, Vicsek 1983). Fractals of lattice type, consisting of sites connected by bonds, are valuable model systems for many theoretical purposes, including the study of phase transitions or modelling transport phenomena. Since dynamical processes are determined by the fractal and spectral (fracton) dimensions (Alexander and Orbach 1982) it is very important to find classes of deterministic fractals in which these dimensions may be easily adjusted to applications through simple parameter changes.

In this letter we investigate new classes of fractal structures: these are generalisations of the d-dimensional Sierpinski gaskets. Here we present their construction, determine their order of ramification and calculate exactly their fractal and spectral dimensions. To evaluate the latter we renormalise the master equations which govern random walks over the structures. We find a large class of fractals in the dimensional range between one and two. The calculations are carried out through matrix-inversion techniques. For several cases we also present closed-form analytic expressions for the renormalisation mappings, which, distinct from the pure Sierpinski case, are not necessarily polynomials.

In the study of fractals the Sierpinski gaskets embedded in d-dimensional Euclidean space (Urysohn 1927, Mandelbrot 1982, Gefen *et al* 1984) are well suited both for analytical considerations (Gefen *et al* 1981, 1984, Rammal 1984, Domany *et al* 1983) and for numerical simulations (Angles d'Auriac *et al* 1983, Blumen *et al* 1983). We note that the gaskets determined by d are members of a more general family of fractals characterised by an additional integer b, ($b \geq 2$). Both parameters determine the generator $G(b, d)$, which is the basic geometrical unit from which the fractal is iteratively constructed (Mandelbrot 1982). The generator $G(b, d)$ is a d-dimensional hypertetrahedron of side length b which is itself filled with b layers of smaller hypertetrahedrons

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of unit side length. In figure 1 we display such geometrical units in two dimensions ($d=2$) for the cases $b=2$, 3 and 4.

Using $G(b, d)$ the fractal is built up iteratively: One obtains the structure at stage $n+1$ by enlarging $G(b, d)$ by b^n and then filling all upward pointing hypertetrahedrons with the stage-n structure. We exemplify this procedure for $G(3, 2)$, for which we display in figure 2 the result of two iterations (i.e. the structure is at stage $n=3$). In the limit $n \to \infty$ the whole fractal is obtained. We remark that the Sierpinski gaskets result from this construction as the special case $b=2$. For general b, $b>2$, the connectivity properties of the fractals differ from the Sierpinski gaskets in that the number $z(\boldsymbol{r}_i)$ of nearest neighbours of each site $\boldsymbol{r}_i$ is not constant over the lattice, whereas for $b=2$ one has $z(\boldsymbol{r}_i)=2d$ for all i.

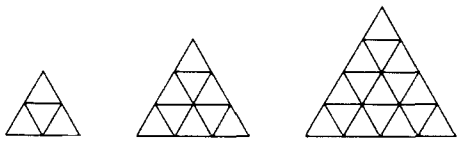

Figure 1. Geometrical units $G(b, d)$ (generators) for different b-values in two dimensions ($d=2$).

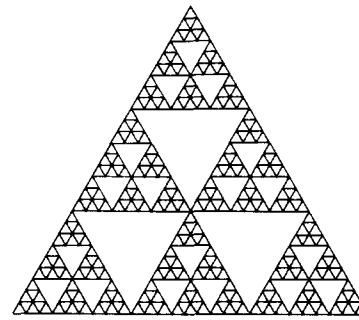

Figure 2. Two-dimensional fractal structure with $b=3$ at the third stage ($n=3$) of the construction.

We start our study of the geometrical properties of the fractals with the connectivity $z(\boldsymbol{r}_i)$. Defining $R_{\max}=\max z(\boldsymbol{r}_i)$, where the $\boldsymbol{r}_i$ vary over the lattice, one has $R_{\max}=bd$ for $b \leq d+1$ and $R_{\max}=d(d+1)$ otherwise, whereas the minimal value of $z(\boldsymbol{r}_i)$ is $2d$. The minimal number of sites which disconnect any interior point from the rest of the fractal is, as in the Sierpinski-gasket case, $R_{\min}=d+1$. $R_{\min}$ and $R_{\max}$ are in fact the orders of ramification for the infinite structure as defined by Mandelbrot (1982) and obey $R_{\max} \geq 2(R_{\min}-1)$ (Urysohn 1927). We note that for $b>d+1$ we have

$$R_{\max}=(d+1)(R_{\min}-1) \tag{1a}$$

while for $b \leq d+1$

$$R_{\max}=b(R_{\min}-1). \tag{1b}$$

For Sierpinski gaskets ($b=2$) (1b) defines the quasihomogeneity (Mandelbrot 1982, ch 14).

A further characterisation of the fractal is provided by the Hausdorff dimension $\bar{d}$. This dimension is related to the density of sites belonging to the fractal (Mandelbrot 1982). Here we have:

$$b^{\bar{d}}=\lim_{n\to\infty}[N(b, d; n+1)/N(b, d; n)] \tag{2}$$

where $N(b, d; n)$ is the number of sites of the structure at stage n. The $N(b, d; n)$ fulfil the recursion relation

$$N(b, d; n+1)=N(b, d; 1)+N(b-1, d; 1)[N(b, d; n)-(d+1)] \tag{3}$$

as may be verified from the construction. From the generators one reads off:

$$N(b, d; 1) = \binom{b+d}{d}$$

where the right-hand side denotes the binomial coefficient. From (2) and (3) $N(b, d; n)$ can be expressed in closed form as:

$$N(b, d; n) = F^n(1 + d/b) - (1 - F^{-1})^{-1}(F^{n-1} - 1)(d - d/b) \tag{4}$$

with $F = \binom{b+d-1}{d}$. For the Sierpinski gasket ($b = 2$) (4) reduces to (Rammal 1984)

$$N(2, d; n) = \tfrac{1}{2}(d+1)[(d+1)^n + 1].$$

However, from (2) and (3) it follows directly that

$$b^{\bar{d}} = N(b-1, d; 1) \tag{5}$$

so that

$$\bar{d} = \ln\binom{b+d-1}{d} \Big/ \ln b.$$

The dynamical properties of a fractal are determined by the spectral (fracton) dimension $\tilde{d}$ (Alexander and Orbach 1982, Rammal and Toulouse 1983); in general this quantity is different from the fractal dimension $\bar{d}$. Under the name 'effective dimensionality', the spectral dimension was already introduced and studied by Dhar (1977). If we denote by $\rho(\omega)$ the density of normal modes of the fractal, one has for low frequencies ω:

$$\rho(\omega) \sim \omega^{\tilde{d}-1}$$

(Dhar 1977, Rammal and Toulouse 1983). The dimension $\tilde{d}$ also appears in diffusion or random walk problems on the fractal. For $\tilde{d} < 2$ the average number of sites S_n visited by a random walker on the lattice during an n-step walk is governed by $\tilde{d}$ (Rammal and Toulouse 1983, Blumen *et al* 1983):

$$S_n \sim n^{\tilde{d}/2}.$$

The higher moments of the distribution of sites visited by the walker also depend on the spectral dimension (Angles d'Auriac *et al* 1983, Blumen *et al* 1983). Furthermore $\tilde{d}$ turns up in the solution of the Schrödinger equation on fractals (Domany *et al* 1983).

In this letter we determine $\tilde{d}$ from the long time behaviour of continuous time random walks on the fractals.

Consider now a nearest-neighbour random walk starting at the origin $\boldsymbol{r}_0$ at time $t = 0$. The probability $P(\boldsymbol{r}_i, t)$ to find the walker at site $\boldsymbol{r}_i$ at time t obeys the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\boldsymbol{r}_i, t) = \sum_{j(i)} [w_{ij} P(\boldsymbol{r}_j, t) - w_{ji} P(\boldsymbol{r}_i, t)] \tag{6}$$

where the sum runs over all $\boldsymbol{r}_j$ that are nearest neighbours to $\boldsymbol{r}_i$ and the w_{ij} are the transition probabilities per unit time from $\boldsymbol{r}_j$ to $\boldsymbol{r}_i$. We specify the w_{ij} through the relation

$$z(\boldsymbol{r}_j) w_{ij} = w = \text{constant} \tag{7}$$

where w is a constant transition rate, and where, as before, $z(\boldsymbol{r}_j)$ is the coordination number of $\boldsymbol{r}_j$. This choice emerges naturally from the discrete-time analogue of the

random walk in which the walker steps at fixed times to one of its $z(\boldsymbol{r}_i)$ nearest neighbours with equal probabilities. As mentioned, in general $z(\boldsymbol{r}_i)$ is not constant over the lattice, so that for $b \geqslant 3$ the transition rates w_{ij} are not symmetric.

Laplace-transforming (6) and using the initial condition $P(\boldsymbol{r}_i, 0) = \delta_{\boldsymbol{r}_i, \boldsymbol{r}_0}$ we get our transformed equation

$$-(\delta_{\boldsymbol{r}_i, \boldsymbol{r}_0}/w) + (1-\alpha)P(\boldsymbol{r}_i, u) = \sum_{j(i)} P(\boldsymbol{r}_j, u)/z(\boldsymbol{r}_j). \tag{8}$$

Here we set $\alpha := -u/w$, which is a dimensionless quantity. Writing (8) in matrix notation, the right-hand side involves the symmetric adjacency matrix of the fractal and a diagonal matrix containing the coordination numbers, while the left-hand side is diagonal.

Equation (8) turns up in many physical fields. As a mechanical analogue the sites are occupied by masses, which are connected along the bonds through springs of strength k. The site dependent transfer rates then translate into site dependent masses $m_j = z(\boldsymbol{r}_j)m$ which are multiples of a unit mass m. The variable $P(\boldsymbol{r}_j, u)$ is the Fourier-transformed transversal displacement of the mass at $\boldsymbol{r}_j$, while α corresponds to mu^2/k. Note, however, that in this case the inhomogeneous term $-\delta_{\boldsymbol{r}_i, \boldsymbol{r}_0}/w$ in (8) is generally missing. The electrical analogue consists of a network where each lattice bond has the resistance R and each lattice site is grounded to a common potential via a site dependent capacitor $C_j = z(\boldsymbol{r}_j)C$. Now $\alpha = -iuRC$ and $P(\boldsymbol{r}_j, u)$ is the Fourier-transformed node potential.

We proceed to analyse (8) and apply a renormalisation procedure, which reduces the number of variables by inverting the iteration process through which the fractals were generated. In one step of the renormalisation procedure one eliminates all interior sites from the generators contained in the gasket, such that the smallest hypertetrahedrons left have side length b instead of 1. Figure 3 illustrates this for the case $b = 3$, $d = 2$. In the following we distinguish notationally between deleted sites $\boldsymbol{d}_i$ and surviving sites $\boldsymbol{s}_i$. Algebraically the removal of the interior sites is performed by expressing the $P(\boldsymbol{d}_i, u)$ in terms of the $P(\boldsymbol{s}_i, u)$ and inserting the results into the equations for the $P(\boldsymbol{s}_i, u)$. For long times the solutions to the original system (8) can be approximated by the solutions of a reduced (albeit formally similar) system of equations involving only the $\boldsymbol{s}_i$, provided the transfer rates and the probabilities are suitably adjusted. The spectral dimension $\tilde{d}$ follows then from the asymptotic behaviour of $P(\boldsymbol{r}_0, t)$, where $\boldsymbol{r}_0$ is the starting site of the walk. For convenience, we choose $\boldsymbol{r}_0$ to be one of the $\boldsymbol{s}_i$, so that $\boldsymbol{r}_0$ is not eliminated in the first decimation step.

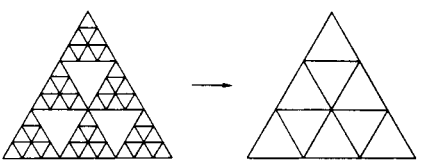

Figure 3. One step of the renormalisation transformation for $b = 3$ and $d = 2$.

We now consider an arbitrary surviving site $\boldsymbol{s}_0$ with coordination number $z(\boldsymbol{s}_0) = ld$ $(2 \leqslant l \leqslant d+1)$ and let $G(b, d)$ be one of the l generators to which $\boldsymbol{s}_0$ belongs. The remaining corners are denoted $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_d$ and the $M = N - d - 1$ inner sites of $G(b, d)$ are labelled $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_M$ beginning with the d nearest neighbours of $\boldsymbol{s}_0$, such that $\boldsymbol{s}_0$, $\boldsymbol{d}_i$ and $\boldsymbol{s}_i$ are collinear $(1 \leqslant i \leqslant d)$. The problem is most clearly formulated in terms of

the normalised probabilities $Q(\boldsymbol{r}_i, u) \equiv P(\boldsymbol{r}_i, u)/z(\boldsymbol{r}_i)$. We collect the $Q(\boldsymbol{r}_i, u)$ into two vectors $\boldsymbol{Q}_1 = (Q(\boldsymbol{d}_1, u), \ldots, Q(\boldsymbol{d}_M, u))$ and $\boldsymbol{Q}_2(Q(\boldsymbol{s}_0, u), \ldots, Q(\boldsymbol{s}_d, u))$.

Considering only the equations for the interior sites $\boldsymbol{d}_i$ we have from (8):

$$[(1-\alpha)\boldsymbol{D} - \boldsymbol{A}_1]\boldsymbol{Q}_1 = \boldsymbol{A}_2\boldsymbol{Q}_2. \tag{9}$$

In this equation $\boldsymbol{D}$ is a diagonal $M \times M$-matrix whose entries are $(\boldsymbol{D})_{ii} = z(\boldsymbol{r}_i)$. The matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are submatrices of the adjacency matrix $\boldsymbol{A}$ of $G(b, d)$. The $M \times M$-submatrix $\boldsymbol{A}_1$ is obtained from $\boldsymbol{A}$ by eliminating the rows and columns corresponding to corners of $G(b, d)$, while the $M \times (d+1)$-submatrix $\boldsymbol{A}_2$ is obtained through elimination of $d+1$ rows corresponding to the corners and M columns corresponding to interior sites. In all our cases $(\boldsymbol{D} - \boldsymbol{A}_1)$ is invertible and we can solve (9) for one of the nearest neighbours of $\boldsymbol{s}_0$, say $\boldsymbol{d}_1$, giving

$$Q(\boldsymbol{d}_1, u) = g(\alpha)Q(\boldsymbol{s}_0, u) + \sum_{i=1}^{d} h_i(\alpha)Q(\boldsymbol{s}_i, u) \tag{10a}$$

where

$$g(\alpha) = [(\boldsymbol{D} - \boldsymbol{A}_1 - \alpha\boldsymbol{D})^{-1}\boldsymbol{A}_2]_{11} \qquad \text{and} \qquad h_i(\alpha) = [(\boldsymbol{D} - \boldsymbol{A}_1 - \alpha\mathrm{D})^{-1}\boldsymbol{A}_2]_{1,i+1}. \tag{10b}$$

We now use the symmetry of $G(b, d)$ under the cyclic group of rotations generated by $\boldsymbol{R}$, where the rotation $\boldsymbol{R}$ of $G(b, d)$ is defined through its action on the corners

$$\boldsymbol{R}\boldsymbol{s}_0 = \boldsymbol{s}_0, \qquad \boldsymbol{R}\boldsymbol{s}_d = \boldsymbol{s}_1, \qquad \boldsymbol{R}\boldsymbol{s}_i = \boldsymbol{s}_{i+1} \qquad \text{for } 1 \leqslant i \leqslant d-1. \tag{11}$$

Operating with $\boldsymbol{R}$ j-times on (10), i.e. using $\boldsymbol{R}^j$, gives

$$Q(\boldsymbol{d}_{j+1}, u) = Q(\boldsymbol{R}^j\boldsymbol{d}_1, u) = g(\alpha)Q(\boldsymbol{s}_0, u) + \sum_{i=1}^{d} h_i(\alpha)Q(\boldsymbol{R}^j\boldsymbol{s}_i, u) \tag{12}$$

and thus we get by summing (12)

$$\sum_{i=1}^{d} Q(\boldsymbol{d}_i, u) = dg(\alpha)Q(\boldsymbol{s}_0, u) + h(\alpha)\sum_{i=1}^{d} Q(\boldsymbol{s}_i, u) \tag{13}$$

where

$$h(\alpha) = \sum_{i=1}^{d} h_i(\alpha).$$

We have thus expressed the $Q(\boldsymbol{d}_i, u)$ for the nearest neighbours of $\boldsymbol{s}_0$ in terms of the $Q(\boldsymbol{s}_i, u)$. The result holds for each of the l generators to which $\boldsymbol{s}_0$ belongs. Equation (8) for $P(\boldsymbol{s}_0, u)$ reads in terms of normalised probabilities

$$-\delta_{\boldsymbol{r}_0, \boldsymbol{s}_0}/w + ld(1-\alpha)Q(\boldsymbol{s}_0, u) = \sum_{k=1}^{l}\sum_{i=1}^{d} Q(\boldsymbol{d}_i^k, u) \tag{14}$$

where we distinguish sites in the different generators by upper indices k, $1 \leqslant k \leqslant l$.

Inserting (13) into (14) the result can be recast as

$$-\delta_{\boldsymbol{r}_0, \boldsymbol{s}_0}/[wh(\alpha)] + ld[1-\phi(\alpha)]Q(\boldsymbol{s}_0, u) = \sum_{k=1}^{l}\sum_{i=1}^{d} Q(\boldsymbol{s}_i^k, u) \tag{15}$$

with

$$\phi(\alpha) = 1 - \{[1 - \alpha - g(\alpha)]/h(\alpha)\}. \tag{16}$$

We see that (14) and (15) are *similar*, and that the effect of the renormalisation consists in replacing w by $wh(\alpha)$ and α by $\phi(\alpha)$. In (16) the coordination number ld has dropped out, which means that $\phi(\alpha)$ is independent of the choice of s_0. This result is due to the judicious choice for the transfer rates w_{ij}.

To proceed further we consider the situation at longer times. In terms of $\alpha = -u/w$ this corresponds to the limit $\alpha \to 0$. Moreover, at longer times the physical situation is characterised by quasi-stationarity inside each generator $G(b, d)$. In equilibrium, say at $t = \infty$, one has from (6) and (7)

$$P(\boldsymbol{r}_i, t)/z(\boldsymbol{r}_i) = P(\boldsymbol{r}_j, t)/z(\boldsymbol{r}_j) = \text{constant} \tag{17}$$

and therefore for $u = 0$:

$$Q(\boldsymbol{r}_i, u = 0) = Q(\boldsymbol{r}_j, u = 0) = \text{constant}. \tag{18}$$

That (18) is indeed a solution to (9) with $\alpha = 0$ is immediately evident by inspection. Inserting (18) into (10) we find

$$1 = g(0) + \sum_{i=1}^{d} h_i(0) = g(0) + h(0). \tag{19}$$

Therefore, with (16) one has $\phi(0) = 0$. By construction $\phi(\alpha)$ is a rational function of α and thus $\phi(\alpha) = \kappa\alpha + O(\alpha^2)$ with $\kappa = \phi'(0)$.

Furthermore, we find $h(0) = N/\kappa$, where $N \equiv N(b-1, d; 1)$ is the number of hypertetrahedrons inside the generator $G(b, d)$. The physical reason for the appearance of N is the requirement of probability conservation: the sum of the interior $P(\boldsymbol{d}_i, t)$ is redistributed on the $P(\boldsymbol{s}_i, t)$. Performing this redistribution for the special case (17) leads immediately to N. We are now able to consider (15) for α around 0. Because in this region we have $\phi(\alpha) = \kappa\alpha$ and $h(\alpha) = N/\kappa$, we find that (15) for the quantities $NQ(s_0, u)$ and the rates w/κ is *identical* to (14). Herewith the renormalisation of our system of equations is completed.

We turn now to the determination of the spectral dimension $\tilde{d}$. For random walks $\tilde{d}$ can be determined from the probability to be at the origin $P(\boldsymbol{r}_0, t)$, since at longer times one has (Alexander and Orbach 1982)

$$P(\boldsymbol{r}_0, t) \sim (1/wt)^{\tilde{d}/2}$$

which renormalised reads

$$NP(\boldsymbol{r}_0, t) \sim (\kappa/wt)^{\tilde{d}/2}$$

and thus

$$N = \kappa^{\tilde{d}/2}, \qquad \text{i.e. } \tilde{d} = 2 \ln N/\ln \kappa. \tag{20}$$

The value of κ can be computed via $\kappa = N/[1 - g(0)]$, while N is given in (5). This procedure is advantageous, since for $g(0)$ only a single row of the matrix $(\boldsymbol{D} - \boldsymbol{A})^{-1}$ is needed, namely (10b).

Symmetry arguments entered the above considerations, and we use them again to reduce the number of equations in (9). We have to analyse only one generator $G(b, d)$, so that we take s_0 as origin, $s_0 = (0, \ldots, 0)$, placing the coordinate axes along the edges of $G(b, d)$, thus $s_1 = (b, 0, \ldots, 0), \ldots, s_d = (0, \ldots, 0, b)$. The set of sites in $G(b, d)$ supports a natural equivalence relation, which carries over to the corresponding probabilities. The equivalence class $\{\boldsymbol{r}_i\}$ of a site $\boldsymbol{r}_i$ are all sites which obtain by

permuting the coordinates of $\boldsymbol{r}_i$. We then call the sum Q_i a class variable:

$$Q_i = \sum_{\boldsymbol{r}_j \in \{\boldsymbol{r}_i\}} Q(\boldsymbol{r}_j, u).$$

Using the equivalence relation $\{\cdot\}$ we treat explicitly the case $b = 3$ for arbitrary d. In this case (9) reduces to a system in at most five class variables for the internal sites of $G(b, d)$. As representatives $\boldsymbol{r}_i$ for the Q_i we take $(1, 0, \ldots, 0)$ for Q_1, $(2, 0, \ldots, 0)$ for Q_2, $(1, 2, 0, \ldots, 0)$ for Q_3, $(1, 1, 0, \ldots, 0)$ for Q_4 and $(1, 1, 1, 0, \ldots, 0)$ for Q_5. Class Q_5 appears only for $d \geq 3$. The coordination number for the representatives of Q_1, Q_2 and Q_3 is $2d$ and for Q_4 and Q_5 it is $3d$. We set $Q_0 = Q(\boldsymbol{s}_0, u)$ for the corner at the origin and Q_6 for the other corners, represented by $(3, 0, \ldots, 0)$. With $\boldsymbol{Q} \equiv (Q_1, \ldots, Q_5)$ and $\boldsymbol{Y} = (-dQ_0, -Q_6, -(d-1)Q_6, 0, 0)$, equation (9) takes the form $\boldsymbol{CQ} = \boldsymbol{Y}$ with:

$$\boldsymbol{C} = \begin{pmatrix} 2d\alpha - d - 1 & 1 & 0 & 2 & 0 \\ 1 & 2d\alpha - 2d & 1 & 2 & 0 \\ 0 & d-1 & 2d\alpha - d - 1 & 2 & 6 \\ d-1 & d-1 & 1 & 3d\alpha - d - 4 & 3 \\ 0 & 0 & d-2 & d-2 & 3d\alpha - 9 \end{pmatrix}. \tag{21}$$

From (21), after some lengthy algebra, the mapping $\phi(\alpha)$ follows

$$\phi(\alpha) = \alpha(d + 2 - 2d\alpha)(4d + 6 - 6d\alpha)^{-1}[(2d + 8 - 6d\alpha)(d + 2 - 2d\alpha) - 3(d-1)]. \tag{22}$$

For Sierpinski gaskets, $b = 2$, one has after symmetrisation only two class variables, and $\phi(\alpha)$ is a polynomial, $\phi(\alpha) = \alpha(d + 3 - 2d\alpha)$. For both $b = 2$ and $b = 3$ the values $\alpha = 0$ and $\alpha = \infty$ are fixed points. For $b = 3$ three additional fixed points and a simple pole at $\alpha = d^{-1} + 2/3$ appear, and we have

$$\kappa = \phi'(0) = (d+2)(2d^2 + 9d + 19)/(4d + 6)$$

which with (5) and (20) gives

$$\tilde{d} = \frac{2 \ln[(d+1)(d+2)/2]}{\ln[(d+2)(2d^2 + 9d + 19)/(4d+6)]}. \tag{23}$$

For $b = 2$ one has $\tilde{d} = 2\ln(d+1)/\ln(d+3)$, so that in both cases $\lim_{d \to \infty} \tilde{d} = 2$.

In figure 4 we display the spectral dimensions $\tilde{d}$ for $b = 2$ and $b = 3$ as a function of d. In both cases the convergence to the limiting value 2 is evident. For fixed d, the values of $\tilde{d}$ for $b = 3$ lie above those for $b = 2$; with increasing d the difference between the $\tilde{d}$-values first increases to a maximum (0.056 at $d = 4$) and then decreases.

To evaluate $\phi(\alpha)$ and $\tilde{d}$ for $b \geq 4$ we proceed numerically and make use of the formulation in terms of class variables. As above one has to solve a system of coupled equations for $G(b, d)$ leading to an expression for the class Q_1 of nearest neighbours of $\boldsymbol{s}_0$. The vector $\boldsymbol{Y}$ in (21) has only three non-zero components. This is due to the fact that the corners $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_d$ are connected to only three classes of sites. The first is the class of nearest neighbours to $\boldsymbol{s}_0$ represented by $(1, 0, \ldots, 0)$. The nearest neighbours of $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_d$ fall into two classes, the first being represented by $(b-1, 0, \ldots, 0)$, the other by $(b-1, 1, 0, \ldots, 0)$. To calculate $\phi(\alpha)$ it is therefore sufficient to compute three elements of $\boldsymbol{C}^{-1}$. For different values $b \geq 3$ and $2 \leq d \leq 6$ we have determined $\phi(\alpha)$ numerically using an algorithm for the generalised eigenvalue problem (Moler and Stewart 1973) and calculating four determinants as functions of α.

The values of $\tilde{d}$ for different d and b are summarised in the table 1. For $b=3$ we recover numerically the exact results, (23). To display the data we plot in figure 5 the spectral dimension as a function of d and b. The $\tilde{d}$-values are more affected by changes in d than in b. In all cases $\tilde{d}=2$ is the upper limit. From our results, with a judicious choice of both b and d, we can adjust $\tilde{d}$ flexibly. We remark that we have further generalised the class of fractals discussed here by combining for fixed d generators with different b-values to form a new generator. We thus obtain a dense set of $\tilde{d}$-values in the interval $1.365 \leqslant \tilde{d} \leqslant 2.0$, thereby allowing us to fix $\tilde{d}$ with arbitrary accuracy.

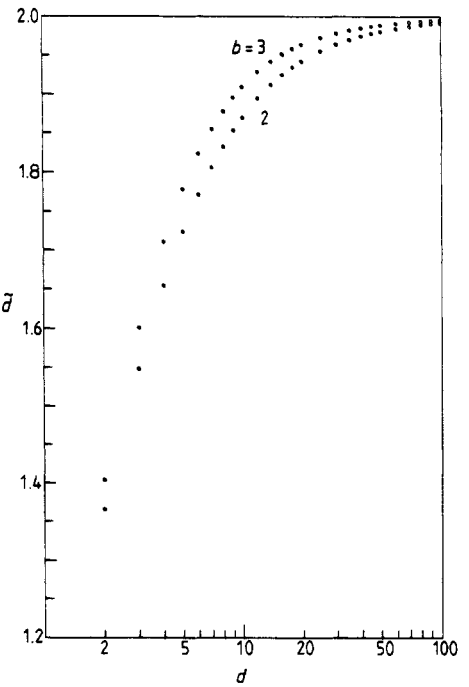

Figure 4. Spectral dimensions $\tilde{d}$ for fractals as a function of the Euclidean dimension d with $b=2$ and $b=3$.

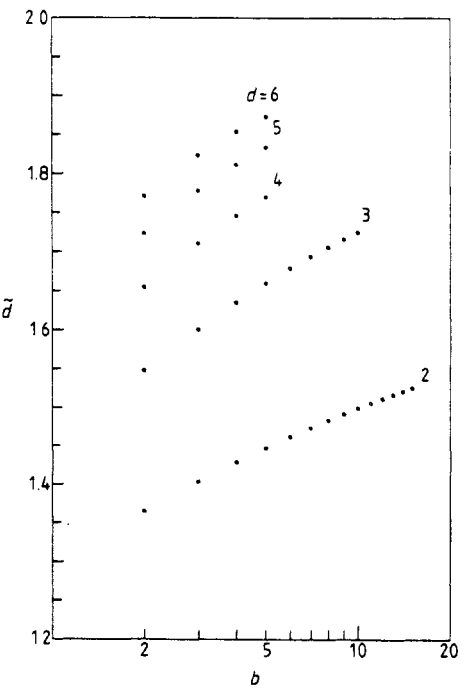

Figure 5. Spectral dimensions $\tilde{d}$ obtained by matrix inversion as a function of b for $1 \leqslant d \leqslant 6$.

Table 1. Values of the spectral dimension $\tilde{d}$ for the fractals constructed-from $G(b, d)$.

b \ d	2	3	4	5	6
2	1.3652	1.5474	1.6542	1.7233	1.7712
3	1.4032	1.6002	1.7104	1.7783	1.8234
4	1.4285	1.6346	1.7456	1.8114	1.8534
5	1.4471	1.6594	1.7700	1.8334	1.8728
6	1.4617	1.6783			
7	1.4734	1.6933			
8	1.4833	1.7056			
9	1.4917	1.7160			
10	1.4990	1.7249			

Summarising, we have discussed a new class of fractals which generalise the Sierpinski gaskets, and have determined their fractal and spectral dimensions exactly. Their renormalisation mappings are rational, not necessarily polynomial functions, and thus lead to new mathematical aspects. The fractals have a finite order of ramification and are of site-and-bond type. Insofar they offer new possibilities for choosing structures with prescribed properties and can serve as useful models in physical applications.

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